# RIEMANNIAN GEOMETRY 2017-2018: SUMMARY

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**Review of smooth manifolds.** Smooth manifolds: definition (local coordinates) and examples (Euclidean space, sphere, regular surfaces in $\mathbb{R}^3$, products, Lie groups).

Smooth maps, diffeomorphisms. Tangent space: definition and basis. Differential of a map. Immersions and submersions: definition and examples (of a map which is an immersion/submersion and of a map which is not).

The tangent bundle and vector fields. Lie bracket of vector fields: definition, properties and examples (a manifold with a pair $X, Y$ such that $[X, Y] = 0$ and a pair $U, V$ such that $[U, V] \neq 0$).

Partitions of unity: definition and example on the sphere subordinated to the stereographic atlas.

**Riemannian metrics and manifolds.** Definition of Riemannian metric. Examples of Riemannian manifolds (Euclidean metric, induced metrics via immersions, hyperbolic metric, products). Existence of metrics on smooth manifolds.

Local and global isometries: definition and examples (of a map which is an isometry and of a map which is not). Length of a curve: definition and example (length of the same curve in $\mathbb{R}^2$ with respect to different metrics).

Isometry group: definition and examples (of the standard spaces). Quotient Riemannian manifolds by proper free actions by discrete groups. Examples of free actions and of non-free actions. Examples of isometric actions and of non-isometric actions.

Left-invariant metrics on Lie groups.

**Connections.** Affine connections. Covariant derivative along a curve. Local coordinates (Christoffel symbols). Parallel vector field and parallel transport.

Connections on Riemannian manifolds: compatible connections and symmetric connections. The Levi-Civita connection: existence and uniqueness (Koszul formula). The Levi-Civita connection on the Euclidean space and on its hypersurfaces.

**Geodesics.** Definition of geodesic. Existence and uniqueness. Definition of the exponential map. Geodesics in the Euclidean space and in the sphere. Definition of the distance function (importance of taking the inf nad not the min) and minimizing properties of geodesics. Examples of geodesics which are minimizing and of geodesics which are non-minimizing.

**Curvature.** Definition of the curvature tensor (following Do Carmo). Properties of the curvature tensor.

Sectional curvature: definition and geometrical interpretation. The curvature tensor on manifolds of constant sectional curvature: identity and idea of the proof.

Jacobi fields: definition and examples. Jacobi fields on manifolds with constant sectional curvature. Characterization of Jacobi fields with $J(0) = 0$ in terms of the exponential map. Conjugate points: definition, relation to the exponential map, examples of manifolds with conjugate points and of manifolds without conjugate points.

**Complete manifolds. Global theorems.** Definition of (geodesically) complete manifolds, examples and non-examples. Statement of the (short version we saw of the) Hopf-Rinow theorem.

Manifolds with curvature $\leq 0$: Cartan-Hadamard theorem. Statement of the theorem, idea of the proof and consequences (examples of manifolds that admit such a metric and of manifolds that do not admit it).

Manifolds with constant curvature: the sphere, the Eculidean plane and the hyperbolic plane with their standard metrics. Idea of the proof of the following facts: the sphere with the standard metric has curvature $\equiv 1$ (isometric immersions), the hyperbolic space has curvature $\equiv -1$ (conformal changes).

Manifolds with curvature $\geq 1$. Bonnet-Myers theorem. Statement of the theorem, idea of the proof and consequences (examples of manifolds that admit such a metric and of manifolds that do not admit it).